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Point-countable bases and k-networks

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Abstract

We present a space with a point-countable base but no point-countable closed k-network.

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Intrinsic characterizations of topological spaces which are the images of metric spaces under quotient maps with separable fibers—that is, quotient s-images of metric spaces—have been given by Hoshina [5] and by Gruenhage, Michael and Tanaka [3]. Both of these conditions are easily seen to hold in k-spaces with a point-countable k-network of closed sets, and we wondered whether this condition might also characterize quotient s-images of metric spaces. But here we present a completely regular space with a uniform base, hence an open-compact image of a metric space [1], which admits no point-countable k-network of closed sets. This answers a question raised in [3]. On the other hand, a regular first-countable space with a point-countable k-network of closed sets has a point-countable base [2].

1. Definition of the space X

Write $\mathbb{R} = \mathbb{Q} \cup \mathbb{P}$, where $\mathbb{Q}$ is the set of rational numbers and $\mathbb{P}$ is the set of irrational numbers. Let

$$G = \mathbb{R} \times \omega \cup \mathbb{P} \cup \{\theta\}.$$

Points in our space X are certain functions from $\omega \times \omega$ to G. Endow $\omega \times \omega$ with the (well-ordered) lexicographical ordering $\leqslant$. Points of X are of type I or type II defined as follows.

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Type I points := the functions from $\omega \times \omega$ to $\mathbb{P}$.

Type II points := the functions x satisfying the following conditions.

(1) There are $i(x) < \omega$ and $0 < j(x) < \omega$ so that $x(i,j) \in \mathbb{P} \times \omega$ if $(i,j) \leqslant (i(x), j(x))$. Define $\beta(x(i,j))$ and $\nu(x(i,j))$ by $x(i,j) = (\beta(x(i,j)), \nu(x(i,j)))$ for $(i,j) \leqslant (i(x), j(x))$.

(2) For $i < i(x)$, there is a $0 < j(x,i) < \omega$ so that $\nu(x(i, j(x,i))) < \nu(x(i,j))$ for all $j \neq j(x,i)$. For $j_1 \leqslant j_2 < \omega$ with $j(x,i) \notin \{j_1, j_2\}$, $\nu(x(i,j_1)) \leqslant \nu(x(i,j_2))$.

(3) For $i = i(x)$, there is a $0 < j(x,i) \leqslant j(x)$ so that $\nu(x(i,j(x,i))) < \nu(x(i,j))$ for all $j \leqslant j(x)$ with $j \neq j(x,i)$. For $j_1 \leqslant j_2 \leqslant j(x)$ with $j(x,i) \notin \{j_1, j_2\}$, $\nu(x(i,j_1)) \leqslant \nu(x(i,j_2))$.

(4) For $j > j(x)$, $x(i(x), j) = \theta$.

(5) For $i > i(x)$, there is a unique $0 < j(x,i) < \omega$ so that $x(i, j(x,i)) \in \mathbb{Q} \times \{0, 1, \ldots, j(x)\}$. Define $\beta(x(i,j(x,i)))$ and $\nu(x(i,j(x,i)))$ by $x(i,j(x,i)) = (\beta(x(i,j(x,i))), \nu(x(i,j(x,i))))$.

(6) For $i > i(x)$ and $j \neq j(x,i)$, $x(i,j) = \theta$.

(7) There is an $M(x) < \omega$ so that $\nu(x(i,j)) \leqslant M(x)$ for all $(i,j) \in \omega \times \omega$ for which $\nu(x(i,j))$ is defined.

We give G a non-Hausdorff topology by providing a countable neighborhood base at each point as follows. For $m < \omega$

$$U\big((q,k),m\big) = \left\{(r,k)\colon\ r \in \mathbb{R} \text{ with } |r-q| < \frac{1}{m+1}\right\}, \quad \text{if } q \in \mathbb{Q} \text{ and } k < \omega;$$

$$U\big((p,k),m\big) = \big\{(p,k)\big\}, \quad \text{if } p \in \mathbb{P} \text{ and } k < \omega;$$

$$U(p,m) = \{p\} \cup \big\{(p,k)\colon\ m \leqslant k < \omega\big\}, \quad \text{if } p \in \mathbb{P};$$

$$U(\theta,m) = \{\theta\} \cup \big\{(r,k)\colon\ r \in \mathbb{R} \text{ and } m \leqslant k < \omega\big\}.$$

We now topologize X by giving a countable neighborhood base at each point as follows. For $x \in X$ and $m,n < \omega$

$$V(x,m,n) = \big\{y \in X\colon\ y(i,j) \in U\big(x(i,j),m\big) \text{ for all } i \leqslant n \text{ and } j < \omega\big\},$$

$$V(x,n) = V(x,n,n).$$

It is easy to see that the space X is Hausdorff and has both a σ-disjoint base and a development. Once we show that X is regular it will follow from [4] that X has a uniform base.

In fact, X is zero-dimensional. Indeed, it is easy to see that type II points have neighborhood bases of clopen sets. We show that type I points do too.

For a type I point x and $0 < n < \omega$, let

$$\begin{aligned} W(x,n) = V(x,n) \cup \bigcup \big\{V\big(y, 1+\nu\big(y(n+1, j(y,n+1))\big)\big), n+1\big)\colon\ & y \text{ is a type II} \\ & \text{point with } i(y) = n,\ \beta\big(y(i,j)\big) = x(i,j) \\ & \text{and } \nu\big(y(i,j)\big) \geqslant n \text{ for all } (i,j) \leqslant \big(n, j(y)\big)\big\}. \end{aligned}$$

We use the following lemmata to show that the $W(x,n)$'s are clopen and form a local base at x.

Lemma. *$W(x,n)$ is open.*

Lemma. $W(x,n) \subset V(x,n-1)$.

Lemma. $\overline{W(x,n)} \subset V(x,n-1)$.

Proof. If $z \in \overline{W(x,n)} \setminus V(x,n-1)$, then $z \in \overline{V(x,n-1)} \setminus V(x,n-1)$, hence z is a type II point with $i(z) = n-1$. Find $m < \omega$ such that

(a) $m > \nu(z(n,j(z,n)))$,

(b) $m \geqslant n$, and

(c) $|x(n,j(z,n)) - \beta(z(n,j(z,n)))| \geqslant 1/(m+1)$.

Now (b) and (c) give $V(z,m) \cap V(x,n) = \emptyset$, so $V(z,m) \cap W(x,n) \neq \emptyset$ implies there is a type II point y with $i(y) = n$ and $\beta(y(i,j)) = x(i,j)$ and $\nu(y(i,j)) \geqslant n$ for all $(i,j) \leqslant (n,j(y))$, and a point

$$t \in V(z,m) \cap V\big(y, 1 + \nu\big(y(n+1, j(y,n+1))\big), n+1\big).$$

As $t \in V(y,0,n+1)$,

$$\begin{aligned}\beta\big(t(n,j(t,n))\big) &= \beta\big(t(n,j(y,n))\big) = \beta\big(y(n,j(y,n))\big)\\ &= x\big(n,j(y,n)\big) = x\big(n,j(t,n)\big).\end{aligned}$$

But $t \in V(z,m)$ and (a) imply $j(t,n) = j(z,n)$; thus

$$\big|\beta\big(t(n,j(t,n))\big) - \beta\big(z(n,j(z,n))\big)\big| < \frac{1}{m+1},$$

and, by (c),

$$\beta\big(t(n,j(t,n))\big) \neq x\big(n,j(z,n)\big) = x\big(n,j(t,n)\big).$$

This proves the lemma. □

Lemma. *$\overline{W(x,n)} \setminus W(x,n)$ contains no type I points.*

Proof. If z is a type I point in $\overline{W(x,n)} \setminus W(x,n)$, let j^* be the least ordinal for which $z(n,j^*) \neq x(n,j^*)$. Pick a $t \in V(z,j^*,n+1) \cap W(x,n)$. Then there is a type II point y with $i(y) = n$ and $\beta(y(i,j)) = x(i,j)$ and $\nu(y(i,j)) \geqslant n$ for all $(i,j) \leqslant (n,j(y))$ so that

$$t \in V\big(y, 1 + \nu\big(y(n+1, j(y,n+1))\big), n+1\big).$$

Now $V(z,0,n+1) \cap V(y,0,n+1) = \emptyset$ if $j^* \leqslant j(y)$, so $j^* > j(y) \geqslant \nu(y(n+1, j(y,n+1)))$. We have $t \in V(z,j^*,n+1)$, which implies $\nu(t(n+1,j(y,n+1))) \geqslant j^*$, while $t \in V(y,0,n+1)$ implies

$$\nu\big(t(n+1, j(y,n+1))\big) = \nu\big(y(n+1, j(y,n+1))\big) < j^*.$$

This proves the lemma. $\square$

Lemma. $\overline{W(x,n)} \setminus W(x,n)$ *contains no type II points.*

Proof. If z is a type II point with $z \in \overline{V(x,n)} \setminus V(x,n)$, then $i(z) = n$ and $\beta(z(i,j)) = x(i,j)$ and $\nu(z(i,j)) \geqslant n$ for all $(i,j) \leqslant (n, j(z))$; hence $z \in W(x,n)$. We now consider a type II point z in the closure of

$$\bigcup\{V\big(y, 1+\nu\big(y(n+1, j(y,n+1))\big), n+1\big)\colon\ y \text{ is a type II point with } i(y) = n \text{ and } \beta\big(y(i,j)\big) = x(i,j) \text{ and } \nu\big(y(i,j)\big) \geqslant n \text{ for all } (i,j) \leqslant (n, j(y))\big\}.$$

We consider three cases.

In case $i(z) > n+1$, there is a type II point y with $i(y) = n$ and $\beta(y(i,j)) = x(i,j)$ and $\nu(y(i,j)) \geqslant n$ for all $(i,j) \leqslant (n, j(y))$ so that

$$V(z, 0, n+1) \cap V\big(y, 1+\nu\big(y(n+1, j(y,n+1))\big), n+1\big) \neq \emptyset.$$

Then $z \in V(y, 1+\nu(y(n+1, j(y,n+1))), n+1)$.

In case $i(z) = n+1$, there is a type II point y with $i(y) = n$ and $\beta(y(i,j)) = x(i,j)$ and $\nu(y(i,j)) \geqslant n$ for all $(i,j) \leqslant (n, j(y))$ and a point

$$t \in V(z, 0, n+1) \cap V\big(y, 1+\nu(y(n+1, j(y,n+1))), n+1\big).$$

Since $t \in V(y, 1+\nu(y(n+1, j(y,n+1))), n+1)$, we have $j(y,n+1) = j(t,n+1) = j(z,n+1) \leqslant j(z)$. It follows that $z \in V(y, 1+\nu(y(n+1, j(y,n+1))), n+1)$.

In case $i(z) = n$, there is a type II point y with $i(y) = n$ and $\beta(y(i,j)) = x(i,j)$ and $\nu(y(i,j)) \geqslant n$ for all $(i,j) \leqslant (n, j(y))$ and a point

$$\begin{aligned} t \in\ & V\big(z, 1+\nu\big(z(n+1, j(z,n+1))\big), n+1\big) \\ & \cap V\big(y, 1+\nu\big(y(n+1, j(y,n+1))\big), n+1\big). \end{aligned}$$

Since $t \in V(z, 0, n) \cap V(y, 0, n)$, if $j(z) \leqslant j(y)$ we get $\beta(z(i,j)) = x(i,j)$ and $\nu(z(i,j)) \geqslant n$ for all $(i,j) \leqslant (n, j(z))$; thus $z \in W(x,n)$. If $j(y) \leqslant j(z)$, then since $j(y,n+1) = j(t,n+1) = j(z,n+1)$ and

$$\nu\big(y(n+1, j(y,n+1))\big) = \nu\big(t(n+1, j(t,n+1))\big) = \nu\big(z(n+1, j(z,n+1))\big),$$

defining y^* from $\omega \times \omega$ to G by

$$y^*(i,j) = \begin{cases} z(i,j) & \text{if } (i,j) = (n+1, j(y,n+1)); \\ y(i,j) & \text{otherwise;} \end{cases}$$

gives a type II point y^* so that $z \in V(y^*, 1+\nu(y^*(n+1, j(y^*, n+1))), n+1) \subset W(x,n)$. This completes the proof of the lemma. $\square$

2. X admits no point-countable closed k-network

Let us introduce some definitions and notation. Given a set D we will say Δ *is an ordered partition of* D if Δ is a finite sequence $\langle S(0), S(1), \ldots, S(A-1)\rangle$ of subsets of

D so that $\{S(a)\colon a < A\}$ is a partition of D. We will say that a function f from D to ω is *increasing on* Δ if f is constant on each $S(a)$, and $f[S(a)] \leqslant f[S(b)]$ if $a \leqslant b < A$. We identify f with the sequence $\langle f[S(0)], f[S(1)], \ldots, f[S(A-1)]\rangle$ in ω.

Proposition 1. *Let $\Delta = \langle S(0), S(1), \ldots, S(A-1)\rangle$ be an ordered partition of a countably infinite set D with $A \geqslant 2$, and let F be the set of all functions from D to ω which are increasing on Δ. Suppose we can write $F = \bigcup\{F(b, A)\colon b \in B\}$ with $0 < |B| < \omega$. For $b \in B$ and $0 < a \leqslant A-1$, set*

$$F(b,a) = \big\{\big\langle k(0), k(1), \ldots, k(a-1)\big\rangle\colon \textit{there are infinitely many } k(a) < \omega \textit{ so that } \big\langle k(0), k(1), \ldots, k(a-1), k(a)\big\rangle \in F(b, a+1)\big\}.$$

Then there is a $b \in B$ so that $F(b,1)$ is infinite.

Proof. If not, pick $k^*(0) > \max(\bigcup\{F(b,1)\colon b \in B\})$. For $0 < a < A$ inductively choose

$$k^*(a) > \max\Big(\big\{k^*(a-1)\big\} \cup \big\{k(a)\colon \big\langle k^*(0), \ldots, k^*(a-1), k(a)\big\rangle \in \bigcup\big\{F(b, a+1)\colon b \in B\big\}\Big).$$

Note that $\langle k^*(0), k^*(1), \ldots, k^*(A-1)\rangle$ is an increasing sequence not in F. $\square$

Suppose $0 < n < \omega$ and for $0 < i \leqslant n$ we have $n < t(i) < s(i) < r(i) < \omega$. For

$$D = \big\{(i,j) \in \omega \times \omega\colon (0,n) < (i,j) \leqslant (n+1, 0)\big\} \setminus \big\{(i, t(i))\colon 0 < i \leqslant n\big\}$$

(where $\omega \times \omega$ is lexicographically ordered) we define an ordered partition

$$\Delta\big(t(n), t(n-1), \ldots, t(1), s(1), r(1), s(2), r(2), \ldots, s(n), r(n)\big)$$

of D by

$$\begin{aligned}
&\Delta\big(t(n), t(n-1), \ldots, t(1), s(1), r(1), s(2), r(2), \ldots, s(n), r(n)\big)\\
&\quad = \Big\langle \big\{(0,j)\colon n < j < \omega\big\}, \big\{(1,j)\colon j < t(1)\big\},\\
&\qquad \big\{(1,j)\colon t(1) < j \leqslant s(1)\big\}, \big\{(1, s(1)+1)\big\}, \big\{(1, s(1)+2)\big\},\\
&\qquad \ldots, \big\{(1, r(1))\big\}, \big\{(1,j)\colon r(1) < j < \omega\big\}, \big\{(2,j)\colon j < t(2)\big\},\\
&\qquad \big\{(2,j)\colon t(2) < j \leqslant s(2)\big\}, \big\{(2, s(2)+1)\big\}, \big\{(2, s(2)+2)\big\},\\
&\qquad \ldots, \big\{(2, r(2))\big\}, \big\{(2,j)\colon r(2) < j < \omega\big\},\\
&\qquad \vdots\\
&\qquad \big\{(n,j)\colon j < t(n)\big\}, \big\{(n,j)\colon t(n) < j \leqslant s(n)\big\}, \big\{(n, s(n)+1)\big\},\\
&\qquad \big\{(n, s(n)+2)\big\}, \ldots, \big\{(n, r(n))\big\},\\
&\qquad \big\{(i,j)\colon (n, r(n)) < (i,j) \leqslant (n+1, 0)\big\}\Big\rangle.
\end{aligned}$$

The object of this section is to prove that our space X has no point-countable k-network of closed sets.

Definition. A collection $\mathcal{R}$ of subsets of a topological space Y is a *k-network* for Y if for each compact set K and open subset $U \supset K$, there is a finite $\mathcal{R}^* \subset \mathcal{R}$ so that $K \subset \bigcup \mathcal{R}^* \subset U$.

Proposition 2 [6,3]. *If Y is a first-countable space with a point-countable k-network $\mathcal{R}$ of closed sets, U is an open subset of Y, and $y \in U$, then there is a finite $\mathcal{R}^* \subset \mathcal{R}$ so that $y \in \operatorname{int}(\bigcup \mathcal{R}^*) \subset \bigcup \mathcal{R}^* \subset U$.*

We begin the proof that our space X has no point-countable k-network $\mathcal{R}$ by choosing a fixed type I point x^* in X and letting $U = V(x^*, 0)$. Proposition 2 would imply that for every type I point $x' \in V(x^*, 0)$ there is an $n(x')$ and a finite $\mathcal{R}^* \subset \mathcal{R}$ so that

$$V\big(x', n(x')\big) \subset \bigcup \mathcal{R}^* \subset V(x^*, 0).$$

We will show that this does not hold.

We need a bit of machinery. Suppose $0 < n < \omega$ and suppose x is a function from $(n+1) \times \omega$ to $\mathbb{P}$ which can be extended to a function x' from $\omega \times \omega$ to $\mathbb{P}$ (i.e., to a type I point x') so that there is a finite subset $\mathcal{R}^*(x) = \{R(b)\colon b \in B(x)\}$ of $\mathcal{R}$ so that

$$V(x', n) \subset \bigcup \mathcal{R}^*(x) \subset V(x^*, 0).$$

Let Σ denote the set of all integer-valued finite sequences σ of length at least 0, but not more than $3n$, which can be extended to a sequence

$$\big\langle t(n), t(n-1), \ldots, t(1), s(1), r(1), s(2), r(2), \ldots, s(n), r(n)\big\rangle$$

with $n < t(i) < s(i) < r(i)$ for each i with $1 \leqslant i \leqslant n$. We pick an $R^*(\sigma)(x) \in \mathcal{R}^*(x)$ for each $\sigma \in \Sigma$ as follows.

We begin with the sequences of length $3n$. For

$$\sigma = \big\langle t(n), t(n-1), \ldots, t(1), s(1), r(1), s(2), r(2), \ldots, s(n), r(n)\big\rangle$$

consider the ordered partition

$$\Delta = \Delta\big(t(n), t(n-1), \ldots, t(1), s(1), r(1), s(2), r(2), \ldots, s(n), r(n)\big)$$

of

$$D = \big\{(i,j) \in \omega \times \omega\colon\ (0,n) < (i,j) \leqslant (n+1, 0)\big\} \setminus \big\{(i, t(i))\colon\ 1 \leqslant i \leqslant n\big\}.$$

For a given $l < \omega$ and a function f from D to $\{m\colon n < m < \omega\}$ which is increasing on Δ, we consider a type II point $y(l, f)(x) \in V(x', n)$ described by

$$y(l,f)(x)(i,j) = \begin{cases} (x^*(i,j), n+1) & \text{if } i = 0,\ j < n; \\ (x^*(i,j), n) & \text{if } i = 0,\ j = n; \\ (x(i,j), n) & \text{if } 1 \leqslant i \leqslant n \text{ and } j = t(i); \\ (x(i,j), f(i,j)) & \text{if } (i,j) \in D \setminus \{(n+1, 0)\}; \\ (e/l, f(i,j)) & \text{if } i = n+1 \text{ and } j = 0; \\ (e/l, 0) & \text{if } i = n+1 \text{ and } j = 1; \\ \theta & \text{if } i = n+1 \text{ and } j > 1; \\ (0,0) & \text{if } i > n+1 \text{ and } j = 1; \\ \theta & \text{if } i > n+1 \text{ and } j \neq 1. \end{cases}$$

(Here e is a particular fixed irrational number.)

Now each $y(l, f)(x) \in R(b)$ for some $b \in B(x)$. Let

$$A = |\Delta| \quad \left(= 3n + 1 + \sum_{i=1}^{n} r(i) - \sum_{i=1}^{n} s(i) \right)$$

and, for each $b \in B(x)$, let $F(b, A)(x)$ be the set of functions f from D to $\{m\colon n < m < \omega\}$ which are increasing on Δ and have $y(l, f)(x) \in R(b)$ for infinitely many $l < \omega$. Adopting the notation of Proposition 1, we select $R^*(\sigma)(x) = R(b)$ for a $b \in B(x)$ so that $F(b, 1)(x)$ is infinite.

We now recursively select $R^*(\sigma)(x) \in \mathcal{R}^*(x)$ for $\sigma \in \Sigma$ of length less than $3n$ so that $R^*(\sigma)(x) = R^*(\widehat{\sigma j})(x)$ for infinitely many $j < \omega$ so that $\widehat{\sigma j} \in \Sigma$. This completes the selection of the $R^*(\sigma)(x)$ for $\sigma \in \Sigma$.

Set $R^*(x) = R^*(\phi)(x)$, where ϕ is the sequence of length 0.

We select some particular members of Σ as follows. Choose

$$\big\langle t(n,0)(x), t(n-1,0)(x), \ldots, t(1,0)(x), s(1,0)(x), r(1,0)(x), s(2,0)(x),$$
$$r(2,0)(x), \ldots, s(n,0)(x), r(n,0)(x) \big\rangle$$

in Σ of length $3n$ so that

$$\begin{aligned} R^*(x) &= R^*\big(\langle t(n,0)(x)\rangle\big)(x) \\ &= R^*\big(\langle t(n,0)(x), t(n-1,0)(x)\rangle\big)(x) \\ &= R^*\big(\langle t(n,0)(x), t(n-1,0)(x), t(n-2,0)(x)\rangle\big)(x) \\ &\ \ \vdots \\ &= R^*\big(\langle t(n,0)(x), t(n-1,0)(x), \ldots, s(n,0)(x), r(n,0)(x)\rangle\big)(x). \end{aligned}$$

Now choose

$$\langle t(n)^*(x), t(n-1,1)(x), t(n-2,1)(x), \ldots, s(1,1)(x), r(1,1)(x), \\ s(2,1)(x), r(2,1)(x), \ldots, s(n-1,1)(x), r(n-1,1)(x)\rangle$$

in Σ of length $3n-2$ so that $t(n)^*(x) > s(n,0)(x)$ and

$$\begin{aligned} R^*(x) &= R^*\big(\langle t(n)^*(x)\rangle\big)(x) \\ &= R^*\big(\langle t(n)^*(x), t(n-1,1)(x)\rangle\big)(x) \\ &= R^*\big(\langle t(n)^*(x), t(n-1,1)(x), t(n-2,1)(x)\rangle\big)(x) \\ &\vdots \\ &= R^*\big(\langle t(n)^*(x), t(n-1,1)(x), \ldots, s(n-1,1)(x), r(n-1,1)(x)\rangle\big)(x). \end{aligned}$$

For $k = 1, 2, \ldots, n$, inductively choose

$$t(n-k+1)^*(x), t(n-k,k)(x), t(n-k-1,k)(x), \\ \ldots, t(1,k)(x), s(1,k)(x), r(1,k)(x), s(2,k)(x), r(2,k)(x), \\ \ldots, s(n-k,k)(x), r(n-k,k)(x)$$

so that

$$\langle t(n)^*(x), t(n-1)^*(x), \ldots, t(n-k+1)^*(x), t(n-k,k)(x), t(n-k-1,k)(x), \\ \ldots, t(1,k)(x), s(1,k)(x), r(1,k)(x), s(2,k)(x), r(2,k)(x), \\ \ldots, s(n-k,k)(x), r(n-k,k)(x)\rangle$$

is a member of Σ of length $3n-2k$ and $t(n-k+1)^*(x) > s(n-k+1,k-1)(x)$ and so that

$$\begin{aligned} R^*(x) &= R^*\big(\langle t(n)^*(x)\rangle\big)(x) \\ &= R^*\big(\langle t(n)^*(x), t(n-1)^*(x)\rangle\big)(x) \\ &= R^*\big(\langle t(n)^*(x), t(n-1)^*(x), t(n-2)^*(x)\rangle\big)(x) \\ &\vdots \\ &= R^*\big(\langle t(n)^*(x), t(n-1)^*(x), \ldots, s(n-k,k)(x), r(n-k,k)(x)\rangle\big)(x). \end{aligned}$$

Note that $R^*(x) = R^*(\langle t(n)^*(x), t(n-1)^*(x), \ldots, t(1)^*(x)\rangle)(x)$.

This completes the construction of our machinery for fixed n with $0 < n < \omega$ and fixed function x from $(n+1)\times\omega$ to $\mathbb{P}$ which can be extended to a type I point x' so that there is a finite subset $\mathcal{R}^*(x)$ of $\mathcal{R}$ so that $V(x',u) \subset \bigcup\mathcal{R}^*(x) \subset V(x^*,0)$.

Now if $0 < n < \omega$ and $\tau = \langle\tau(1), \tau(2), \ldots, \tau(n)\rangle$ is a n-length finite sequence of integers with $\tau(i) > n$ for $1 \leqslant i \leqslant n$; if $1 \leqslant m < n$; and if $R \in \mathcal{R}$; then define:

$$H(\tau) = \big\{x\colon x \text{ is a function from } (n+1)\times\omega \text{ to } \mathbb{P} \text{ so that } x \text{ can be extended} \\ \text{to a type I point } x' \text{ for which there is a finite subset } \mathcal{R}^* \text{ of } \mathcal{R} \\ \text{so that } V(x',n) \subset \textstyle\bigcup\mathcal{R}^* \subset V(x^*,0) \text{ and so that} \\ \langle t(1)^*(x), t(2)^*(x), \ldots, t(n)^*(x)\rangle = \tau\big\}.$$

$$H(n,\tau) = \bigl\{z\colon z \text{ is a function from } \bigl\{(i,j) \in \omega \times \omega\colon\ (i,j) < (n,\tau(n))\bigr\} \text{ to } \mathbb{P} \text{ for which there is a second category set of } p \in \mathbb{P} \text{ so that } \widehat{zp} \text{ can be extended to a member of } H(\tau)\bigr\}.$$

$$H(m,\tau) = \bigl\{z\colon z \text{ is a function from } \bigl\{(i,j) \in \omega \times \omega\colon\ (i,j) < (m,\tau(m))\bigr\} \text{ to } \mathbb{P} \text{ for which there is a second category set of } p \in \mathbb{P} \text{ so that } \widehat{zp} \text{ can be extended to a member of } H(m+1,\tau)\bigr\}.$$

$$H(\tau,R) = \{x \in H(\tau)\colon\ R^*(x) = R\}.$$

$$H(n,\tau,R) = \bigl\{z \in H(n,\tau)\colon \text{There is a second category set of } p \in \mathbb{P} \text{ so that } \widehat{zp} \text{ can be extended to a member of } H(\tau,R)\bigr\}.$$

$$H(m,\tau,R) = \{z \in H(m,\tau)\colon \text{There is a second category set of } p \in \mathbb{P} \text{ so that } \widehat{zp} \text{ can be extended to a member of } H(m+1,\tau,R)\bigr\}.$$

If z is a function from $\{(i,j) \in \omega \times \omega\colon\ (i,j) < (I,J)\}$ to $\mathbb{P}$ and $p \in \mathbb{P}$, then by $\widehat{zp}$ we mean the function from $\{(i,j) \in \omega \times \omega\colon\ (i,j) \leqslant (I,J)\}$ to $\mathbb{P}$ given by

$$\widehat{zp}\,(i,j) = \begin{cases} z(i,j) & \text{if } (i,j) < (I,J), \\ p & \text{if } (i,j) = (I,J). \end{cases}$$

With our notation in place we now turn to the proof that X admits no point-countable k-network of closed sets. The heart of the matter is contained in the next two lemmata and the proof is assembled in Proposition 3.

Lemma. *If $1 \leqslant m \leqslant n < \omega$ and $\tau = \langle \tau(1), \tau(2), \ldots, \tau(n)\rangle$ is an n-length sequence of integers with $\tau(i) > n$ for $1 \leqslant i \leqslant n$, then $H(m,\tau) = \bigcup\{H(m,\tau,R)\colon\ R \in \mathcal{R}\}$.*

Proof. First we consider $m = n$. If $z \in H(n,\tau)$, then there is a second category set C of $p \in \mathbb{P}$ so that $\widehat{zp}$ can be extended to a $x(p) \in H(\tau)$. For each such $x(p)$,

$$R^*\bigl(x(p)\bigr) = R^*\bigl(\langle t(n,0)(x(p)), t(n-1,0)(x(p)), \ldots, s(n,0)(x(p))\rangle\bigr)\bigl(x(p)\bigr).$$

Find $r_n(x(p)) \geqslant \tau(n) = t(n)^*(x(p)) > s(n,0)(x(p))$ so that

$$R^*\bigl(\langle t(n,0)(x(p)), t(n-1,0)(x(p)), \ldots, s(n,0)(x(p)), r_n(x(p))\rangle\bigr)\bigl(x(p)\bigr) = R^*(x(p)).$$

Since there are only countably many possibilities for the sequence $\langle t(n,0)(x(p)), t(n-1,0)(x(p)), \ldots, s(n,0)(x(p)), r_n(x(p))\rangle$ for the $p \in C$, we may assume they coincide for all $p \in C$.

For $p \in C$, let us index the ordered partition

$$\begin{aligned}&\Delta\big(t(n,0)(x(p)),t(n-1,0)(x(p)),\dots,s(n,0)(x(p)),r_n(x(p))\big)\\&\quad=\big\{S(a)\colon\ a<A\big\}.\end{aligned}$$

Note that $\{(n,\tau(n))\}=S(a_0)$ for some a_0 with $1<a_0<A$.

Recall that for each $p\in C$, $R^*(x(p))=R(b)$ for some $b\in B(x(p))$ for which $F(b,1)(x(p))$ is infinite. Thus $F(b,a_0)(x(p))$ is infinite and we may select a function $f(x(p))\in F(b,a_0)(x(p))$. For each $k<\omega$, we may extend $f(x(p))$ to a function $f_k(x(p))\in F(b,A)(x(p))$ so that $f_k(x(p))$ is increasing on $\{S(a)\colon\ a<A\}$ and $f_k(x(p))[S(a)]\geqslant k$ for all a with $a_0\leqslant a<A$. Since there are but countably many possibilities for the $f(x(p))$, we may assume they coincide for all $p\in C$ and write $f(z)$ for $f(x(p))$.

The type II point described by

$$c(i,j)=\begin{cases}(x^*(i,j),n+1) & \text{if } i=0 \text{ and } j<n,\\ (x^*(i,j),n) & \text{if } i=0 \text{ and } j=n,\\ (z(i,j),n) & \text{if } 1\leqslant i\leqslant n \text{ and } j=t(i,0),\\ \theta & \text{if } i=n \text{ and } j\geqslant\tau(n),\\ \theta & \text{if } i\geqslant n+1 \text{ and } j\neq 1,\\ (0,0) & \text{if } i\geqslant n+1 \text{ and } j=1,\\ (z(i,j),[f(z)](i,j)) & \text{otherwise;}\end{cases}$$

is in $\operatorname{cl}(R^*(x(p)))$ for each $p\in C$. Indeed, if $k<\omega$ find $l(k)>k$ so that $y(l(k),f_k(x(p)))(x(p))\in R^*(x(p))$ and note that the sequence $\{y(l,(k),f_k(x(p)))(x(p))\colon k<\omega\}$ converges to c. $\mathcal{R}$ being a point-countable collection of closed sets, there is an $R^*(z)\in\mathcal{R}$ and a second category set of $p\in\mathbb{P}$ so that $R^*(x(p))=R^*(z)$. That is, $z\in H(n,\tau,R^*(z))$, and hence the lemma holds if $m=n$.

Now let us suppose $1\leqslant m<n$, and assume that the lemma holds for $m+1$, $m+2,\dots,n$. Let $z\in H(m,\tau)$. There is a second category set of $p_m\in\mathbb{P}$ so that $\widehat{zp_m}$ can be extended to a $z_{m+1}(p_m)\in H(m+1,\tau,R^*(z_{m+1}(p_m)))$. For each such p_m there is a second category set of $p_{m+1}\in\mathbb{P}$ so that $\widehat{z_{m+1}(p_m)p_{m+1}}$ can be extended to a $z_{m+2}(p_m,p_{m+1})\in H(m+2,\tau,R^*(z_{m+1}(p_m)))$. Continuing, we eventually get that for each such $p_m,p_{m+1},\dots,p_{n-1}$, there is a second category set of $p_n\in\mathbb{P}$ so that $\widehat{z_n(p_m,p_{m+1},\dots,p_{n-1})p_n}$ can be extended to an $x(p_m,p_{m+1},\dots,p_n)\in H(\tau,R^*(z_{m+1}(p_m)))$. For each $x(p_m,p_{m+1},\dots,p_n)$ we have

$$\begin{aligned}R^*\big(z_{m+1}(p_m)\big)&=R^*\big(x(p_m,p_{m+1},\dots,p_n)\big)\\&=R^*\big(\big\langle t(n,0)(x(p_m,p_{m+1},\dots,p_n)),t(n-1,0)(x(p_m,p_{m+1},\dots,p_n)),\\&\qquad\dots,s(n,0)(x(p_m,p_{m+1},\dots,p_n)),\\&\qquad r(n,0)(x(p_m,p_{m+1},\dots,p_n))\big\rangle\big)(x(p_m,p_{m+1},\dots,p_n)).\end{aligned}$$

Find an $r_m(x(p_m, p_{m+1}, \ldots, p_n)) \geqslant \tau(m) = t(m)^*(x(p_m, p_{m+1}, \ldots, p_n)) > s(m, n-m)(x(p_m, p_{m+1}, \ldots, p_n))$ so that

$$\begin{aligned} R^*\big(\big\langle & t(n)^*(x(p_m, p_{m+1}, \ldots, p_n), t(n-1)^*(x(p_m, p_{m+1}, \ldots, p_n)), \ldots, \\ & t(m+1)^*(x(p_m, p_{m+1}, \ldots, p_n)), t(m, n-m)(x(p_m, p_{m+1}, \ldots, p_n)), \ldots, \\ & t(1, n-m)(x(p_m, p_{m+1}, \ldots, p_n)), s(1, n-m)(x(p_m, p_{m+1}, \ldots, p_n)), \\ & r(1, n-m)(x(p_m, p_{m+1}, \ldots, p_n)), \ldots, s(m, n-m)(x(p_m, p_{m+1}, \ldots, p_n)), \\ & r_m(x(p_m, p_{m+1}, \ldots, p_n))\big\rangle\big)(x(p_m, p_{m+1}, \ldots, p_n)) \end{aligned}$$

is equal to $R^*(x(p_m, p_{m+1}, \ldots, p_n))$. Then find integers

$$\begin{aligned} & s_{m+1}(x(p_m, p_{m+1}, \ldots, p_n)), r_{m+1}(x(p_m, p_{m+1}, \ldots, p_n)), \\ & s_{m+2}\big(x(p_m, p_{m+1}, \ldots, p_n)\big), r_{m+2}\big(x(p_m, p_{m+1}, \ldots, p_n)\big), \\ & \quad \ldots, s_n\big(x(p_m, p_{m+1}, \ldots, p_n)\big), r_n\big(x(p_m, p_{m+1}, \ldots, p_n)\big) \end{aligned}$$

so that

$$\begin{aligned} R^*\big(\big\langle & t(n)^*\big(x(p_m, p_{m+1}, \ldots, p_n)\big), t(n-1)^*(x(p_m, p_{m+1}, \ldots, p_n)), \ldots, \\ & t(m+1)^*(x(p_m, p_{m+1}, \ldots, p_n)), t(m, n-m)(x(p_m, p_{m+1}, \ldots, p_n)), \ldots, \\ & t(1, n-m)(x(p_m, p_{m+1}, \ldots, p_n)), s(1, n-m)(x(p_m, p_{m+1}, \ldots, p_n)), \\ & r(1, n-m)(x(p_m, p_{m+1}, \ldots, p_n)), \ldots, s(m, n-m)(x(p_m, p_{m+1}, \ldots, p_n)), \\ & r_m(x(p_m, p_{m+1}, \ldots, p_n)), s_{m+1}(x(p_m, p_{m+1}, \ldots, p_n)), \\ & r_{m+1}(x(p_m, p_{m+1}, \ldots, p_n)), \ldots, s_n(x(p_m, p_{m+1}, \ldots, p_n)), \\ & r_n(x(p_m, p_{m+1}, \ldots, p_n))\big\rangle\big)(x(p_m, p_{m+1}, \ldots, p_n)) \end{aligned}$$

is equal to $R^*(x(p_m, p_{m+1}, \ldots, p_n))$. By rechoosing our second category sets above we may assume that the $3n$-length sequence

$$\big\langle t(n)^*(x(p_m, p_{m+1}, \ldots, p_n)), \ldots, r_n(x(p_m, p_{m+1}, \ldots, p_n))\big\rangle$$

just constructed is identical for each $\langle p_m, p_{m+1}, \ldots, p_n\rangle$, and call it σ.

Let us index the ordered partition $\Delta(\sigma) = \{S(a)\colon a < A\}$. There is an a_0 with $1 < a_0 < A$ so that $\{(m, \tau(m))\} = S(a_0)$.

As $R^*(x(p_m, p_{m+1}, \ldots, p_n)) = R(b)$ for some $b \in B(x(p_m, p_{m+1}, \ldots, p_n))$ for which $F(b, 1)(x(p_m, p_{m+1}, \ldots, p_n))$ is infinite, we have that $F(b, a_0)(x(p_m, p_{m+1}, \ldots, p_n))$ is infinite. Let us select a function

$$f\big(x(p_m, p_{m+1}, \ldots, p_n)\big) \in F(b, a_0)\big(x(p_m, p_{m+1}, \ldots, p_n)\big).$$

For each $k < \omega$ we may extend $f(x(p_m, p_{m+1}, \ldots, p_n))$ to a function

$$f_k\big(x(p_m, p_{m+1}, \ldots, p_n)\big) \in F(b, A)\big(x(p_m, p_{m+1}, \ldots, p_n)\big)$$

so that $f_k(x(p_m, p_{m+1}, \ldots, p_n))$ is increasing on $\{S(a)\colon a < A\} = \Delta(\sigma)$ and $f_k(x(p_m, p_{m+1}, \ldots, p_n))[S(a)] \geqslant k$ for all a with $a_0 \leqslant a < A$. Since there are but countably many possibilities for $f(x(p_m, p_{m+1}, \ldots, p_n))$, we may assume they coincide for all $\langle p_m, p_{m+1}, \ldots, p_n\rangle$ and write $f(z)$ for $f(x(p_m, p_{m+1}, \ldots, p_n))$.

For each fixed $\langle p_m, p_{m+1}, \dots, p_{n-1}\rangle$ the set of p_n's is second category and hence the closure of the set of p_n's contains a rational number $q_n(p_m, p_{m+1}, \dots, p_{n-1})$. We may assume the $q_n(p_m, p_{m+1}, \dots, p_{n-1})$'s coincide and write $q_n(z) = q_n(p_m, p_{m+1}, \dots, p_{n-1})$. Likewise, for each fixed $\langle p_m, p_{m+1}, \dots, p_{n-2}\rangle$ the set of p_{n-1}'s is second category and hence its closure contains a rational number $q_{n-1}(p_m, p_{m+1}, \dots, p_{n-2})$. Again we may assume the $q_{n-1}(p_m, p_{m+1}, \dots, p_{n-2})$'s coincide and write $q_{n-1}(z) = q_{n-1}(p_m, p_{m+1}, \dots, p_{n-2})$. Similarly we find $q_{n-2}(z), q_{n-3}(z), \dots, q_{m+1}(z)$.

The type II point c given by

$$c(i,j) = \begin{cases} (x^*(i,j), n+1) & \text{if } i = 0 \text{ and } j < n, \\ (x^*(i,j), n) & \text{if } i = 0 \text{ and } j = n, \\ (z(i,j), n) & \text{if } 1 \leqslant i \leqslant m \text{ and } j = t(i, n-m), \\ \theta & \text{if } i = m \text{ and } j \geqslant \tau(i), \\ (q_i(z), n) & \text{if } m < i \leqslant n \text{ and } j = \tau(i), \\ \theta & \text{if } m < i \leqslant n \text{ and } j \neq \tau(i), \\ \theta & \text{if } i \geqslant n+1 \text{ and } j \neq 1, \\ (0,0) & \text{if } i \geqslant n+1 \text{ and } j = 1, \\ (z(i,j), [f(z)](i,j)) & \text{otherwise;} \end{cases}$$

is in $\mathrm{cl}(R^*(z_{m+1}(p_m))) = \mathrm{cl}(R^*(x(p_m, p_{m+1}, \dots, p_n)))$ for each p_m. Indeed, if $k < \omega$ find a $p_{m+1}(k)$ in the (remaining) second category set so that $|p_{m+1}(k) - q_{m+1}(z)| < 1/(k+1)$. Then for $\langle p_m, p_{m+1}(k)\rangle$ find $p_{m+2}(k)$ in the appropriate second category set so that $|p_{m+2}(k) - q_{m+2}(z)| < 1/(k+1)$. Likewise find $p_{m+1}(k), p_{m+2}(k), \dots, p_n(k)$ so that $|p_i(k) - q_i(z)| < 1/(k+1)$ for $m < i \leqslant n$. Then find $l(k) > k$ so that $y(l(k), f_k(x(p_m, p_{m+1}(k), \dots, p_n(k)))(x(p_m, p_{m+1}(k), \dots, p_n(k))))$ is in

$$R^*\bigl(x(p_m, p_{m+1}(k), \dots, p_n(k))\bigr) = R^*\bigl(z_{m+1}(p_m)\bigr).$$

The sequence

$$\bigl\{y\bigl(l(k), f_k(x(p_m, p_{m+1}(k), \dots, p_n(k)))\bigr)\bigl(x(p_m, p_{m+1}(k), \dots, p_n(k))\bigr)\colon\ k < \omega\bigr\}$$

converges to c. Thus since $R^*(z_{m+1}(p_m))$ is closed, $c \in R^*(z_{m+1}(p_m))$ for each p_m. As $\mathcal{R}$ is point-countable, there is a second category set of p_m's so that the $R^*(z_{m+1}(p_m))$'s coincide, say $R^*(z) = R^*(z_{m+1}(p_m))$. Thus $z \in H(m, \tau, R^*(z))$, and the lemma is proved. $\square$

Lemma. *If $1 \leqslant n < \omega$ and $\tau = \langle \tau(1), \tau(2), \dots, \tau(n)\rangle$ is an n-length sequence of integers with $\tau(i) > n$ for $1 \leqslant i \leqslant n$ then $H(1, \tau) = \emptyset$.*

Proof. If $H(1,\tau) \neq \emptyset$, say $z_1 \in H(1, \tau, R^*)$ for an $R^* \in \mathcal{R}$, then there is a second category set of $p_1 \in \mathbb{P}$ so that $\widehat{z_1 p_1}$ can be extended to a $z_2(p_1) \in H(2, \tau, R^*)$. For each such p_1 there is a second category set of $p_2 \in \mathbb{P}$ so that $\widehat{z_2(p_1)p_2}$ can be extended to a $z_3(p_1,p_2) \in H(3,\tau,R^*)$. Continuing we eventually get that for each such $\langle p_1,p_2,\ldots,p_{n-1}\rangle$ there is a second category set of $p_n \in \mathbb{P}$ so that $\widehat{z_n(p_1,p_2,\ldots,p_{n-1})p_n}$ can be extended to an $x(p_1,p_2,\ldots,p_n) \in H(\tau,R^*)$. For each such $\langle p_1,p_2,\ldots,p_n\rangle$ we have

$$\begin{aligned}R^* &= R^*\big(x(p_1,p_2,\ldots,p_n)\big)\\ &= R^*\big(\big\langle t(n)^*(x(p_1,p_2,\ldots,p_n)), t(n-1)^*(x(p_1,p_2,\ldots,p_n)),\ldots,\\ &\qquad t(1)^*(x(p_1,p_2,\ldots,p_n))\big\rangle\big)\big(x(p_1,p_2,\ldots,p_n)\big)\\ &= R^*(\tau)\big(x(p_1,p_2,\ldots,p_n)\big).\end{aligned}$$

For each appropriate $\langle p_1,p_2,\ldots,p_n\rangle$ we can find a $3n$-length sequence $\sigma(p_1,p_2,\ldots,p_n) \in \Sigma$ which extends τ so that $R^* = R^*(\sigma(p_1,p_2,\ldots,p_n))(x(p_1,p_2,\ldots,p_n))$.

By rechoosing our second category sets we may assume that the $\sigma(p_1,p_2,\ldots,p_n)$'s coincide for the various $\langle p_1,p_2,\ldots,p_n\rangle$'s.

As $R^* = R(b)$ for some $b \in B(x(p_1,p_2,\ldots,p_n))$ for which $F(b,1)(x(p_1,p_2,\ldots,p_n))$ is infinite, we can find for each $k < \omega$ a function $f_k(x(p_1,p_2,\ldots,p_n))$ which is increasing on the ordered partition $\Delta(\sigma(p_1,p_2,\ldots,p_n))$ so that $f_k(x(p_1,p_2,\ldots,p_n))[S] \geqslant k$ for all $S \in \Delta(\sigma(p_1,p_2,\ldots,p_n))$.

For each fixed $\langle p_1,p_2,\ldots,p_{n-1}\rangle$, the set of p_n's is second category and hence the closure of the p_n's contains a rational number $q_n(p_1,p_2,\ldots,p_{n-1})$. We may assume the $q_n(p_1,p_2,\ldots,p_{n-1})$'s coincide and write q_n. Further, for each fixed $\langle p_1,p_2,\ldots,p_{n-2}\rangle$, the set of p_{n-1}'s is second category and hence the closure of the p_{n-1}'s contains a rational $q_{n-1}(p_1,p_2,\ldots,p_{n-2})$; again we may assume the $q_{n-1}(p_1,p_2,\ldots,p_{n-2})$'s coincide and write q_{n-1}. Likewise find $q_n,q_{n-1},\ldots,q_1$.

Consider the type II point c given by

$$c(i,j) = \begin{cases} (x^*(i,j), n+1) & \text{if } i=0 \text{ and } j<n;\\ (x^*(i,j), n) & \text{if } i=0 \text{ and } j=n;\\ (q_i, n) & \text{if } 1 \leqslant i \leqslant n \text{ and } j=\tau(i);\\ (0,0) & \text{if } i \geqslant n+1 \text{ and } j=1;\\ \theta & \text{otherwise.}\end{cases}$$

To see that $c \in R^*$, choose for each $k < \omega$ a $p_1(k)$ in the appropriate second category set so that $|p_1(k) - q_1| < 1/(k+1)$. For this $p_1(k)$ find an appropriate $p_2(k)$ so that $|p_2(k) - q_2| < 1/(k+1)$. Likewise find $p_1(k),p_2(k),\ldots,p_n(k)$ so that $|p_i(k) - q_i| < 1/(k+1)$ for $1 \leqslant i \leqslant n$. Then find $l(k) > k$ so that $y(l(k), f_k(x(p_1(k),p_2(k),\ldots,p_n(k))))(x(p_1(k),p_2(k),\ldots,p_n(k))) \in R^*$. Since R^* is closed and the sequence

$$\big\{y\big(l(k), f_k(x(p_1(k),p_2(k),\ldots,p_n(k)))\big)(x(p_1(k),p_2(k),\ldots,p_n(k)))\colon\ k<\omega\big\}$$

converges to c, we have $c \in R^* \setminus V(x^*, 0) = R^* \setminus U$. This contradicts the original selection of the $R(b)$'s. The lemma is proved. □

Proposition 3. *X has no point-countable k-network of closed sets.*

Proof. We adopt the previous notation. Recall that x^* is a fixed type I point in X and $U = V(x^*, 0)$. $\mathcal{R}$ is the proposed point-countable closed k-network. For every $x \in U$, there is a finite $\mathcal{R}^* \subset \mathcal{R}$ and an integer n so that $V(x, n) \subset \bigcup \mathcal{R}^* \subset U$.

Let T_n be the set of all n-length finite sequence τ of integers with $\tau(i) > n$ for $1 \leqslant i \leqslant n$.

Consider the type I point x' defined as follows. If either $i = 0$ or $j = 0$, then $x'(i,j) = x^*(i,j)$. If we have defined $x'(i,j)$ for $(i,j) < (I,J)$ with $J \neq 0$, then choose $x'(I,J) \in \mathbb{P}$ so that the function z from $\{(i,j) \in \omega \times \omega\colon (i,j) \leqslant (I,J)\}$ to $\mathbb{P}$ defined by

$$z(i,j) = x'(i,j) \quad \text{for all } (i,j) \leqslant (I,J),$$

cannot be extended to a member of $H(I+1, \tau)$ for any $\tau \in T_n$ with $n > I$ and $\tau(I) = J$; nor can it be extended to a member of $H(\tau)$ for any $\tau \in T_I$ and $\tau(I) = J$. This is possible since we need only to choose $x'(I,J) \in \mathbb{P}$ avoiding a first category set.

Now $x' \in V(x^*, 0)$. If $V(x', n) \subset \bigcup \mathcal{R}^* \subset V(x^*, 0)$ for a finite $\mathcal{R}^* \subset \mathcal{R}$, then if x is the restriction of x' to $(n+1) \times \omega$ we have $x \in H(\langle t(1)^*(x), t(2)^*(x), \ldots, t(n)^*(x)\rangle)$, which conflicts with our selection of $x'(n, t(n)^*(x))$. □

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